Office hours today: 2-4 PM

Continue Treeless Theorem...

If $G$ is acyclic, but if any edge is added to $G$ (joining existing non-adjacent vertices), then the resulting graph contains a unique cycle.

Proof: See P. 1174 of text for a proof. We showed above in lemmas 1, 5, 6.
Def: A directed graph (digraph) is a pair $G = (V, E)$ where $V \neq \emptyset$ and $E \subseteq V \times V$.

\[ \begin{array}{c}
\bullet \\
\rightarrow \\
x \quad \rightarrow \\
y \\
(\text{origin}) \quad \text{terminus}
\end{array} \]
\[ V = \{ x, y, u, v \} \]
\[ E = \{ (x, y), (u, x), (v, y), (v, u), (x, v) \} \]

\[ \text{in-degree: } \]
\[ \text{id}(x) = 1 \]
\[ \text{out-degree: } \]
\[ \text{od}(x) = 2 \]

\[ (x, y) \neq (y, x) \]
\[ (x, x) \]

\[ (x, y) \neq (y, x) \]
\[ (x, x) \]
Handshaking Lemma for digraphs:

\[ \sum_{x \in V} id(x) = \sum_{x \in V} od(x) = |E| \]

Two digraphs \( G_1, G_2 \) are said to be isomorphic if there exists a bijection (isomorphism)

\[ \phi : V(G_1) \rightarrow V(G_2) \]

satisfying:

\[ (x, y) \in E(G_1) \iff (\phi(x), \phi(y)) \in E(G_2) \]
Similar definitions for:

- directed walk
- trail
- path
- cycle
- lengths of above

Let $x, y \in V(G)$. We say $y$ is reachable from $x$ if $G$ contains a directed $x-y$ path.
A digraph $G$ is called strongly connected if for all $x, y \in V(G)$, $y$ is reachable from $x$ and $x$ is reachable from $y$.

More generally: A subset $S \subseteq V(G)$ is called strongly connected if for all $x, y \in S$: $y$ reachable from $x$ and $x$ reachable from $y$. 
Defn
$S \subseteq V(G)$ is called a strongly connected component of $G$ iff

(i) $S$ is strongly connected

(ii) $S$ is maximal w.r.t. (i)

Ex

\[ \begin{array}{ccc}
\text{strong components:} \\
\{x, y, z, u\} \\
\end{array} \]
Representations

- Incidence matrix
- Adjacency matrix
- Adjacency list

Let $G = (V, E)$, $|V| = n$, $|E| = m$

Incidence matrix: $n \times m$

Rows $\leftrightarrow$ vertices
Columns $\leftrightarrow$ edges.

Let $V = \{x_1, x_2, \ldots, x_n\}$

$E = \{e_1, e_2, \ldots, e_m\}$
\[ I(G) = (I_{ij}) \]

\[ I_{ij} = \begin{cases} 1 & \text{if } i \text{ incident with } e_j \\ 0 & \text{otherwise} \end{cases} \]

\[ I_{ij} = \begin{cases} 1 & \text{if } i \text{ terminus of } e_j \\ -1 & \text{if } i \text{ origin of } e_j \\ 0 & \text{otherwise} \end{cases} \]

Example:

\[ I = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \]
$I = \begin{pmatrix}
-1 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & -1
\end{pmatrix}$

**Adjacency Matrix** $n \times n$

- Rows $\leftrightarrow$ Vertices
-_cols $\leftrightarrow$ Vertices

$V = \{ x_1, x_2, \ldots, x_n \}$

$A(G) = (A_{i,j})$
**Undirected:**

\[
A_{ij} = \begin{cases} 
1 & \text{if } x_i \text{ adjacent to } x_j \\
0 & \text{otherwise}
\end{cases}
\]

**Directed:**

\[
A_{ij} = \begin{cases} 
1 & (x_i, x_j) \in E \\
0 & \text{otherwise}
\end{cases}
\]

Ex.

\[
A = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0
\end{pmatrix}
\]
Example:

- Adjacency List
  - Array of lists

\[ A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \]

- undirected