Ex. (DFS)

```
1  2  3
\   \   |
\    \  |
\     \|
\      |
4  5  6  7
```

<table>
<thead>
<tr>
<th>adj</th>
<th>d</th>
<th>f</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>8</td>
<td>n</td>
</tr>
<tr>
<td>2</td>
<td>x</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>x</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>x</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>x</td>
<td>13</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>12</td>
<td>6</td>
</tr>
</tbody>
</table>

DFS Forest:

```
1  3
   \\|
   \  |
2   5
```

4  6  7
<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>d</th>
<th>l</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>11</td>
<td>16</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>12</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>5</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>14</td>
<td>15</td>
<td>3</td>
</tr>
</tbody>
</table>

Ex. 

---

Forest.

---

```
1      3
  \
2      4
  \
5      8
  \
6      7
  \
  ```
Runtime:

- \( n = |V|, m = |E| \)
  - load 1-3 cost: \( \Theta(n) \)
  - load 5-7, disregarding 7: \( \Theta(n) \)
  - DFS line 7 \( \Rightarrow \) visit line 6
    - are executed exactly \( n \) times.
  - Visit line 3-6 is executed a total
    - \( \sum_{x \in V} |\text{adj}(x)| \) times:
      - \( \sum_{x \in V} |\text{adj}(x)| = \begin{cases} 2m & \text{undirected} \\ m & \text{directed} \end{cases} \)
    - \( = \Theta(m) \)
  - all other \( a_i \)s are const. cost,
    - so total cost is: \( \Theta(n+m) \).
from previous example: form a parenthesis string

'(' = discover event

')' = finish event

time: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

\[
\begin{array}{cccccccc}
( & ( & ( & ( & ( & ) & ) & ) & ) & ) & ) & ) & ) & ) \\
\hline
5 & 7 & 6 & 4 & 8 & 3 \\
2 & & & & & & & & & & & & & \\
1 & & & & & & & & & & & & & \\
\end{array}
\]
Theorem (Parentheses)

Let \( x, y \in V \) and \( d[x] \leq d[y] \). Then exactly one of the following holds.

1. \( d[x] < d[y] < f \) \( \forall f \) \( \neq t \)
   \( (\quad ) \quad (\quad ) \)

or

2. \( d[x] < d[y] < f \) \( \forall f \) \( \neq t \)
   \( (\quad ) \quad (\quad ) \)

\( d[x] \land t \)
\( d[y] \land s \)
\( t \land s \land f \land j \)

\( \text{impossible} \)

(2)
Remarks

(2) is equivalent to saying:

- y is discovered while x is gray
- y is a descendant of x in some tree

(1) holds only when y is not a descendant of x, so either

- x, y are cousins in same tree
- x, y lie in different trees
Theorem (White Path)

Let $x, y \in V$. Then $y$ is a descendant of $x$ (i.e., (2) holds) if at time $dl\times 1$ $G$ contains an $x-y$ path consisting entirely of white vertices.

Classification of edges

1. **Tree edges**: belong to $G_p$
2. **Back edges**: join vertex to ancestor
3. **Forward edges**: join an ancestor to a descendant (other than child).
4. **Cross edges**: cousin to cousin or tree to tree.
Note: in undirected case, there is no distinction between (2) and (3). Both categories are called back edges in this case.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{a} & \textbf{f} & \textbf{t} \\
\hline
1 & 1 & 8 \\
2 & 2 & 7 \\
3 & 3 & 4 \\
4 & 5 & 6 \\
5 & 9 & 14 \\
6 & 10 & 11 \\
7 & 12 & 13 \\
\hline
\end{tabular}
\end{table}

tree: \((1, 2) (2, 3) (2, 4) (5, 6) (5, 7)\)
back: \((3, 1)\)
forward: \((1, 4)\)
cross: \((7, 6) (5, 4) (6, 4)\)
23.4 Topological Sort

Define a digraph in called **acyclic** iff it contains no directed cycles.

Example:

- [Acyclic Digraph]
- [Not Acyclic Digraph]

**Lemma**

A digraph $G$ is acyclic iff DFS$(G)$ yields no back edges.
Proof.

Equivalently we have:

A digraph $G$ contains a directed cycle if and only if DFS($G$) produces a back edge.

($\Leftarrow$) Obvious. (Travel down a line of descendancy, then up the back edge.)

($\Rightarrow$) Suppose $G$ contains a directed cycle, call it $C$. 

(\begin{tikzpicture}[scale=0.8]
    \node (x) at (0,0) {$x$};
    \node (y) at (1,0) {$y$};
    \node (c) at (0,-1) {$C$};
    \draw[->] (x) to (y);
    \draw[->] (y) to (x);
    \draw[dashed] (c) circle (0.5); % dashed circle
\end{tikzpicture})
Let $y$ be the first vertex on $C$ to be discovered, and let $x$ come immediately before $y$ on $C$.

So at time $d[y]$ there exists a white $y-x$ path in $G$, namely $C$ (without edge $(x,y)$).

Let $x$ be the first vertex in $C$ such that $x$ is a desc. of $y$ in some tree.

$(x,y)$ is a back edge.