CMPS 101
Homework Assignment 3

1. Exercise 1 from the induction handout.
   Prove that for all $n \geq 1$: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Do this twice:
   a. Using form IIa of the induction step.
   b. Using form IIb of the induction step.

2. Exercise 2 from the induction handout.
   Define $S(n)$ for $n \in \mathbb{Z}^+$ by the recurrence:
   $$S(n) = \begin{cases} 
   0 & \text{if } n = 1 \\
   S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2
   \end{cases}$$
   Prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg n)$.

3. Let $T(n)$ be defined by the recurrence formula:
   $$T(n) = \begin{cases} 
   1 & \text{if } n = 1 \\
   T(\lfloor n/2 \rfloor) + n^2 & \text{if } n \geq 2
   \end{cases}$$
   Show that $\forall n \geq 1$: $T(n) \leq \frac{4}{3} n^2$, and hence $T(n) = O(n^2)$. (Hint: follow Example 3 on page 3 of the induction handout.)

4. Let $T(n)$ be defined by the recurrence formula:
   $$T(n) = \begin{cases} 
   2 & \text{if } n = 1, 2 \\
   9T(\lfloor n/3 \rfloor) + 1 & \text{if } n \geq 3
   \end{cases}$$
   Show that $\forall n \geq 1$: $T(n) \leq 3n^2 - 1$, and hence $T(n) = O(n^2)$. (Hint: emulate Example 4 on page 4 of the induction handout. I. Base: check the two cases $n = 1$, and $n = 2$. II. Induction step: show that for all $n \geq 3$, if for any $k$ in the range $1 \leq k < n$ we have $T(k) \leq 3k^2 - 1$, then $T(n) \leq 3n^2 - 1$.)

5. Let $g(n)$ be an asymptotically non-negative function. Prove that $o((g(n)) \cap \Omega(g(n)) = \emptyset$. 
